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Design Approximation Problems for Linear-Phase Nonrecursive Digital Filters

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The topic of this paper is the study of four real, linear, possibly constrained minimum norm approximation problems, which arise in connection with the design of linear-phase nonrecursive digital filters and are distinguished by the type of used trigonometric approximation functions. In the case of unconstrained minimax designs these problems are normally solved by the Parks–McClellan algorithm, which is an application of the second algorithm of Remez to these problems and which is one of the most popular tools in filter design. In this paper the four types of approximation problems are investigated for all L^p and l^p norms, respectively. It is especially proved that the assumptions for the Remez algorithm are satisfied in all four cases, which has been claimed, but is not obvious for three of them. Furthermore, results on the existence and uniqueness of solutions and on the convergence and the rate of convergence of the approximation errors are derived.

1. INTRODUCTION

Digital filters are used in many electronic devices in order to transform a digital signal into a digital signal with requested properties, where primarily (and here) the characteristic properties of the filter in the frequency domain are considered. The purpose of filter design is to generate filters which fulfill ideal prescriptions for these properties as well as possible.

The *frequency response of a nonrecursive* or *finite impulse response (FIR) filter* is given by

$$H_N(h, \omega) := \sum_{k=0}^{N-1} h_k \, e^{-ik\omega}, \qquad \omega \in \mathbb{R},$$

where $h := (h_0, \dots, h_{N-1})^T$ is a vector in $\mathbb{R}^N$ and $i := \sqrt{-1}$. Especially, the frequency response of a *linear-phase FIR filter* has the form

$$H_N(h, \omega) = A_N(x, \omega) \exp\{-i\varphi_N(\omega)\}, \qquad \omega \in \mathbb{R}, \tag{1}$$

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where h and x are connected by linear relations and $A_N(x, \cdot)$ and φ_N are real-valued functions which depend on the *filter length* N and on each other. Furthermore, the function $A_N(x, \cdot)$, which is denoted as the *amplitude response* of the filter, equals

$$A_N(x, \omega) := v(\omega) \sum_{k=0}^{M-1} x_k c_k(\omega), \qquad c_k(\omega) := \cos(k\omega),$$

where v and M are given in one of the following four ways:

$$v(\omega) := 1, \qquad M := (N+1)/2, \qquad N \text{ odd}, \tag{2}$$

$$v(\omega) := \cos(\omega/2), \qquad M := N/2, \qquad N \text{ even}, \tag{3}$$

$$v(\omega) := \sin(\omega), \qquad M := (N-1)/2, \qquad N \text{ odd}, \tag{4}$$

$$v(\omega) := \sin(\omega/2), \qquad M := N/2, \qquad N \text{ even}. \tag{5}$$

Hence four different types of linear-phase filters with real coefficients exist. Details on this can be found, for example, in [13] and [21].

For a design problem, a set of *passbands* $\Omega_{\mathscr{D}}^P \subseteq [0, \pi]$ and *stopbands* $\Omega_{\mathscr{D}}^S \subseteq [0, \pi]$ has to be specified, where typically passbands and stopbands are those frequency intervals for which the corresponding frequency parts of the signal ideally shall pass the filter without any change and shall be suppressed completely by the filter respectively. Thus $\Omega_{\mathscr{D}}^P$ and $\Omega_{\mathscr{D}}^S$ are disjoint sets, and each of them is the union of finitely many disjoint closed intervals in $[0, \pi]$. The union $\Omega_{\mathscr{D}} := \Omega_{\mathscr{D}}^P \cup \Omega_{\mathscr{D}}^S$ of all bands is denoted as the *design frequency domain*. It is a subset of $[0, \pi]$ and usually includes the frequencies 0 and π.

Furthermore, for the design of a filter, a *desired ideal frequency response* D of the filter is to be prescribed as a continuous complex-valued function on $\Omega_{\mathscr{D}}$ which is different from zero on all of $\Omega_{\mathscr{D}}^P$ and vanishes on $\Omega_{\mathscr{D}}^S$. Then, for a nonempty closed subset Ω of $\Omega_{\mathscr{D}}$, one design problem is to approximate D by $H_N(h, \cdot)$ on Ω with respect to some L^p or l^p norm, where possibly the error $D - H_N(h, \cdot)$ is weighted on Ω and additional requirements on the filter coefficient vector h may have to be satisfied in the form of equality and/or inequality constraints. Note that the norm in the minimum norm problem may be defined on a proper subset Ω of $\Omega_{\mathscr{D}}$ only, while constraints on $H_N(h, \cdot)$ may, for example, be defined on the remaining set $\Omega_{\mathscr{D}} \setminus \Omega$. Typical choices for Ω are $\Omega := \Omega_{\mathscr{D}}$, $\Omega := \Omega_{\mathscr{D}}^P$, and $\Omega := \Omega_{\mathscr{D}}^S$. It is assumed here for simplicity and given typically in practice that Ω is either the union of finitely many disjoint closed intervals in $[0, \pi]$ or a finite subset thereof.

Besides the linear complex problem of frequency response approximation, there exist three other central (nonlinear) problems in filter design: to approximate a desired *magnitude response* $|D(\omega)|$, to simultaneously approximate a desired magnitude response $|D(\omega)|$ and *phase response* $\arg(D(\omega))$, and to simultaneously approximate a desired magnitude response $|D(\omega)|$ and *group delay response* $-\frac{d}{d\omega}\arg(D(\omega))$ on proper subsets of $\Omega_{\mathscr{D}}$ by the corresponding functions for the frequency response of the filter (e.g., [5, 13, 15]). For linear-phase filters the phase response and hence also the group delay response of the filter are not changeable by the filter coefficients h_k so that it is reasonable to consider only desired phase responses which equal the phase response of the filter. Therefore, for linear-phase filters (*nonlinear-phase FIR filters* are investigated in [23] and [22]), one has zero phase response and group delay errors and one can write the desired frequency response as

$$D(\omega) = A(\omega)\exp\{-i\varphi_N(\omega)\}, \qquad \omega \in \Omega_{\mathscr{D}}, \tag{6}$$

where A is a real-valued continuous function on $\Omega_{\mathscr{D}}$ denoted as the *desired amplitude response* and φ_N from (1), which is related to the phase response, is a real-valued affine-linear function (see [13]). Moreover, since D vanishes on $\Omega_{\mathscr{D}}^S$ and since by (1) and (6) one gets

$$\begin{aligned}
&2\mathrm{Re}\{D(\omega)\}\ \mathrm{Re}\{H_N(h,\omega)\} + 2\mathrm{Im}\{D(\omega)\}\ \mathrm{Im}\{H_N(h,\omega)\}\\
&\quad = 2\mathrm{Re}\{D(\omega)\,\overline{H_N(h,\omega)}\} = 2\mathrm{Re}\{|D(\omega)|\,e^{-i\varphi_N(\omega)}\,|H_N(h,\omega)|\,e^{i\varphi_N(\omega)}\}\\
&\quad = 2\,|D(\omega)|\,|H_N(h,\omega)|, \qquad \omega \in \Omega_{\mathscr{D}}^P,
\end{aligned}$$

the frequency and magnitude response errors are identical on $\Omega_{\mathscr{D}}$ in this case, i.e., one has

$$|D(\omega) - H_N(h,\omega)|^2 = \big||D(\omega)| - |H_N(h,\omega)|\big|^2, \qquad \omega \in \Omega_{\mathscr{D}}.$$

Thus, owing to the (1) and (6) and to the identity $|\exp\{-i\varphi_N(\omega)\}| = 1$, for linear-phase filters the four mentioned types of approximation problems reduce to one real linear minimum norm problem. If $C(\Omega)$ is the space of all real-valued continuous functions on Ω equipped with the L^p norm

$$\|f\|_{\Omega,p} := \begin{cases} \left\{\int_\Omega |f(\omega)|^p\,d\omega\right\}^{1/p} & \text{if } 1 \leqslant p < \infty,\\ \max_{\omega\in\Omega} |f(\omega)| & \text{if } p = \infty. \end{cases} \tag{7}$$

when Ω is the union of closed intervals and with the l^p norm

$$\|f\|_{\Omega,p} := \begin{cases} \left\{\sum_{\omega\in\Omega} |f(\omega)|^p\right\}^{1/p} & \text{if } 1 \leqslant p < \infty,\\ \max_{\omega\in\Omega} |f(\omega)| & \text{if } p = \infty, \end{cases} \tag{8}$$

when Ω is a finite set, this problem reads

$$\lambda_{p,M} := \min_{x \in K_M} \left\| W \left(A - v \sum_{k=0}^{M-1} x_k c_k \right) \right\|_{\Omega, p}, \tag{9}$$

where v and M are defined by one of the four different choices (2)–(5), $W \in C(\Omega)$ is a positive weight function, and $K_M \subseteq \mathbb{R}^M$ is a proper closed set of parameters. In practice K_M typically is defined by finitely many equality and/or (in)finitely many inequality constraints for continuous functions.

Applications and the solution of problem (9) have been studied extensively for the least-squares norm ($p=2$) and the maximum norm ($p=\infty$), mainly for the unconstrained case $K_M := \mathbb{R}^M$. Among the suggested algorithms, the most popular one is the Parks–McClellan algorithm [14], which is an application of the second algorithm of Remez to the unconstrained problem (9) for $p=\infty$ and a discrete set Ω. The Parks–McClellan algorithm also has been generalized in [4] and [6] (what does not seem to be known much) in order to include the treatment of certain relevant constraint sets $K_M \subseteq \mathbb{R}^M$. For further methods and applications of problem (9) in case of the maximum norm, see especially [13, 16, 21, 26, 28]. Applications and approaches to problem (9) for the least-squares norm are found in [1, 11, 13, 16, 27, 29].

It is easily seen and was observed, e.g., in [13] that the functions $vc_0, \ldots, vc_{M-1}$ generate a *Haar space* (cf. Definition 2.1) if v and M are chosen as in (2) and Ω consists of at least M points (see Lemma 2.1). Hence, convergence of the Remez algorithm for the unconstrained maximum norm problem (9) is especially ensured in this case. (See [3] where the Remez algorithm and the alternant theorem, on which it is based, are stated for an arbitrary closed subset of an interval as needed here, while in the usually given reference [2] both are formulated for a closed interval only.) In [13, p. 87], it is furthermore said that application of the Remez algorithm to (9) for the choices (3), (4), or (5) would be "straightforward," while these selections are not considered at all in other references, as, e.g., in [6]. A result of this paper is that, although the Haar condition is not satisfied in these cases when, e.g., 0 and π belong to Ω, convergence of the Remez algorithm can nevertheless be guaranteed, where the proof of this result is nontrivial (see Theorem 2.2). Thus, with our analysis here, we in particular believe that we close a gap in the filter design theory.

To our knowledge problem (9) has not been investigated systematically from a mathematical point of view. In Section 2 we study the existence and uniqueness of a solution of this problem for the different choices (2)–(5) and for all p-norms, and we especially discuss the characterization of such solution by the alternation condition in case of the maximum norm and the absence of constraints. In Section 3 we derive results on the convergence and the rate of convergence of the sequence of approximation errors $\{\lambda_{p,M}\}_{M\in\mathbb{N}}$. We conclude the paper in Section 4 with some remarks.

2. EXISTENCE, UNIQUENESS, AND CHARACTERIZATION OF SOLUTIONS

We study now the linear real approximation problem

$$\lambda_{p,M} := \min_{x\in K_M} \left\| W\left(A - v\sum_{k=0}^{M-1} x_k c_k\right)\right\|_{\Omega,p} \tag{10}$$

for each of the choices (2)–(5) of v and M. For that we make the following assumptions, where in this section $|\Lambda|$ means the cardinality of some set Λ.

ASSUMPTION 2.1. (i) *$K_M \subseteq \mathbb{R}^M$ is nonempty and closed.*

(ii) *If $|\Omega|<\infty$, one has $|\Omega|\geqslant M+1$ for* (2), (3), *and* (5) *and $|\Omega|\geqslant M+2$ for* (4).

(iii) *One has $A\notin \mathrm{span}\{vc_0, ..., vc_{M-1}\}$ on Ω.*

The set K_M is a real parameter set here. Design problems with complex coefficients have to be studied in the framework of nonlinear-phase filters. (See [23]. We only know of one such example for a linear-phase filter given in [20].)

By straightforwardly extending the classical existence theorem for the unconstrained linear minimum norm problem (e.g., [2, p. 20]) to the constrained case considered here, we arrive at the following theorem, where $\lambda_{p,M}>0$ follows from Assumption 2.1(iii).

THEOREM 2.1. *Under Assumption* 2.1 *problem* (10) *has a solution with $\lambda_{p,M}>0$ for each of the choices* (2)–(5).

We next need to assume that the function A vanishes at 0 and π if the value of the approximating function $v\sum_{k=0}^{M-1} x_k c_k$ equals zero there.

ASSUMPTION 2.2. *One has $A(0)=0$ for* (4) *and* (5) *if $0\in\Omega_{\mathscr{D}}$ and $A(\pi)=0$ for* (3) *and* (4) *if $\pi\in\Omega_{\mathscr{D}}$.*

For this section, the set $\Omega_{\mathscr{D}}$ in Assumption 2.2 can be replaced by the set Ω. However, in the subsequent section we have to relate the set K_M to a subset of the space $C(\Omega_{\mathscr{D}})$ (see Assumption 3.2) and hence to distinguish between both sets.

The adaptations of the Remez algorithm by Parks and McClellan [14] and Grenez [6] for the case $p=\infty$ make essential use of the subsequent definition and result.

DEFINITION 2.1. M continuous functions on a compact set $\Lambda \subseteq \mathbb{R}$ with $|\Lambda| \geqslant M$ are said to generate a M-dimensional *Haar space* or to satisfy the *Haar condition* on Λ if each nonzero linear combination of these functions has at most $M-1$ zeros in Λ.

LEMMA 2.1. *The functions $c_0, \ldots, c_{M-1}$ generate an M-dimensional Haar space on each closed subset of $[0, \pi]$ which contains at least M distinct points.*

Proof. As is well known, for given coefficients x_k there exist y_k and, conversely, for y_k there exist x_k, $k=0, \ldots, M-1$, such that

$$\sum_{k=0}^{M-1} x_k \cos(k\omega) = \sum_{k=0}^{M-1} y_k \cos^k(\omega). \tag{11}$$

Use of $z := \cos(\omega)$ shows that each linear combination of $c_0, \ldots, c_{M-1}$ has at most $M-1$ zeros on each closed subset of $[0, \pi]$ with at least M distinct points. ∎

Furthermore, the following result is of importance in this connection.

LEMMA 2.2. *Let Λ be a closed subset of $[0, \pi]$ with $|\Lambda| \geqslant M$. The functions $vc_0, \ldots, vc_{M-1}$ generate an M-dimensional Haar space on Λ when Λ does not contain π in case* (3)*, 0 and π in case* (4)*, and 0 in case* (5)*. They do not form a Haar space on Λ otherwise.*

Proof. v is positive on $[0, \pi)$ for (3), on $(0, \pi)$ for (4) resp. on $(0, \pi]$ for (5) so that the first part of the lemma follows from Lemma 2.1. By use of (11) and the transformation $z := \cos(\omega)$, moreover, a function $g := \sum_{k=0}^{M-1} x_k c_k$ can be given which has $M-1$ zeros in $(0, \pi)$ so that vg has at least M zeros on each upper set as defined in the lemma. ∎

Lemma 2.1 implies that, for (2), problem (10) has a unique solution and convergence of the second algorithm of Remez is ensured (e.g., [2, 19]). By Lemma 2.2 this is also true for (3)–(5) when Ω, for example, is a closed subset of $(0, \pi)$, but both are not evident when, as it is typical in filter design, Ω is a closed subset of $[0, \pi]$ which encloses 0 and π. Therefore it is proposed, for example, in [21, p. 126], to replace Ω by $\Omega \setminus \{\omega \in \{0, \pi\} \mid v(\omega) = 0\}$ and to extract v from the parentheses in (10). Such action is probably also meant when application of the Remez algorithm to problem (10) with (3), (4), or (5) is denoted as "straightforward" in [13, p. 87] or not considered at all as, e.g., in [6].

The Remez algorithm, however, requires a closed approximation domain so that the described procedure implies that Ω in (10) is replaced by a proper closed subset of Ω. We remark in this connection that the implementation of the Parks–McClellan algorithm [10; 21, p. 139], as typically in filter design, computes function values of the error function in (10) only on a set $\Upsilon \cap \Omega$ with a "dense grid" Υ in $[0, \pi]$ (see also Section 4 in this respect) so that exclusion of the boundary points of Ω with zero error (under Assumption 2.2) indeed leads to a closed subset where the Haar condition is satisfied and hence convergence of the Remez algorithm is guaranteed. (Such a posteriori discretization of the problem has the same effect as the a priori discretization obtained if Ω in (10) is replaced by the finite set $\Upsilon \cap \Omega$.)

We show next that the Remez algorithm can be applied directly to problem (10) for each of the four filter types and every closed subset of $[0, \pi]$, since, although the Haar condition then may not be satisfied for (3)–(5), the problem still has the needed properties which normally are concluded from the validity of the Haar condition. First, we note the subsequent observation. (See [2] for the definitions of the *alternation condition* and *strong unicity*.)

Remark 2.1. By Lemma 2.2 the functions $vc_0, ..., vc_{M-1}$ with v defined by (3), (4), or (5) generate a *weak Chebyshev space* on $[0, \pi]$. (See, e.g., [3] or [12] where, however, results are given only for an interval and not for an arbitrary compact set in $\mathbb{R}$.) The linear maximum norm approximation problem for such space may have more than one solution. In case it has only one, this is characterized by the alternation condition [12, pp. 88–89].

THEOREM 2.2. *Let Assumptions* 2.1 *and* 2.2 *be fulfilled and let* $p = \infty$ *and* $K_M := \mathbb{R}^M$. *Then, for* (2)–(5), *problem* (10) *has a strongly unique solution which is characterized by the alternation condition.*

Proof. For (2) the requested properties follow from Lemma 2.1 [2]. We consider now case (4) and assume that Ω includes 0 and/or π since the result would follow from Lemma 2.2 otherwise. For simplicity we let both 0 and π belong to Ω. The proofs for the case that either 0 or π belong to Ω and for the choices (3) and (5) then become evident.

We let $\mathscr{U} \subseteq C(\Omega)$ be the space generated by $u_k := vc_k$, $k = 0, ..., M-1$, where v is given by (4). By Lemma 2.1, each $u \in \mathscr{U}$ has at most $M+1$ zeros on Ω including those at 0 and π. Moreover, by Assumption 2.1, Ω consists of at least $M+2$ distinct points in this case so that u possesses a unique representation as a linear combination of $u_0, \ldots, u_{M-1}$. For each nonempty closed subset Γ of $[0, \pi]$, we define the problem

$$P[\Gamma]: \qquad \zeta(\Gamma) := \min_{u \in \mathscr{U}} \|W(A-u)\|_{\Gamma, \infty}. \tag{12}$$

Obviously $P[\Gamma]$ has a solution [2, p. 20] and $\zeta(\Omega) = \lambda_{\infty, M}$. Without loss of generality we can assume $W \equiv 1$ on Ω, since otherwise we could redefine A and u by WA and Wu, respectively.

We now let $\Omega_j := [\varepsilon_j, \pi - \varepsilon_j] \cap \Omega$ where $\{\varepsilon_j\}_{j \in \mathbb{N}}$ is a zero sequence of positive numbers. By $|\Omega| \geqslant M+2$ we can assume that ε_j, $j \in \mathbb{N}$, is sufficiently small such that Ω_j contains at least M distinct points, i.e., more points than each $u \in \mathscr{U}$ has zeros on Ω_j. As a consequence of Lemma 2.2, problem $P[\Omega_j]$, $j \in \mathbb{N}$, therefore possesses a unique solution $\hat{u}_j \in \mathscr{U}$. For $\hat{u}_j$ and each solution $\hat{u} \in \mathscr{U}$ of $P[\Omega]$ we especially have

$$\zeta(\Omega_j) \leqslant \|A - \hat{u}\|_{\Omega_j, \infty} \leqslant \|A - \hat{u}\|_{\Omega, \infty} = \zeta(\Omega) \leqslant \|A - \hat{u}_j\|_{\Omega, \infty}. \tag{13}$$

We next observe that

$$\lim_{j \to \infty} \|f\|_{\Omega_j, \infty} = \|f\|_{\Omega, \infty}, \qquad f \in C(\Omega). \tag{14}$$

By a result of [8] therefore $j_0 \in \mathbb{N}$ and $\eta > 0$ exist such that

$$\|u\|_{\Omega, \infty} \leqslant \eta \, \|u\|_{\Omega_j, \infty}, \qquad u \in \mathscr{U}, \quad j \geqslant j_0. \tag{15}$$

Hence we obtain

$$\begin{aligned} \|\hat{u}_j\|_{\Omega, \infty} &\leqslant \eta \, \|\hat{u}_j\|_{\Omega_j, \infty} \leqslant \eta(\|A - \hat{u}_j\|_{\Omega_j, \infty} + \|A\|_{\Omega_j, \infty}) \\ &\leqslant 2\eta \, \|A\|_{\Omega_j, \infty} \leqslant 2\eta \, \|A\|_{\Omega, \infty}, \qquad j \geqslant j_0. \end{aligned}$$

Consequently there exists a subsequence of $\{\hat{u}_j\}_{j \in \mathbb{N}}$ which converges to some $\tilde{u} \in \mathscr{U}$ in the norm $\|\cdot\|_{\Omega, \infty}$. Without loss of generality we can assume that this is the total sequence. By the estimates

$$\begin{aligned} &|\zeta(\Omega_j) - \|A - \tilde{u}\|_{\Omega, \infty}| \\ &\quad \leqslant |\|A - \hat{u}_j\|_{\Omega_j, \infty} - \|A - \tilde{u}\|_{\Omega_j, \infty}| + |\|A - \tilde{u}\|_{\Omega_j, \infty} - \|A - \tilde{u}\|_{\Omega, \infty}| \\ &\quad \leqslant \|\hat{u}_j - \tilde{u}\|_{\Omega, \infty} + |\|A - \tilde{u}\|_{\Omega_j, \infty} - \|A - \tilde{u}\|_{\Omega, \infty}|, \end{aligned} \tag{16}$$

the convergence of $\{\hat{u}_j\}_{j \in \mathbb{N}}$ and (14) entail $\lim_{j \to \infty} \zeta(\Omega_j) = \|A - \tilde{u}\|_{\Omega, \infty}$. Thus, by (13), $\tilde{u}$ is a solution of $P[\Omega]$ and $\lim_{j \to \infty} \zeta(\Omega_j) = \zeta(\Omega)$.

Obviously there is a $j_1 \geqslant j_0$ such that $\zeta(\Omega)/2 \leqslant \zeta(\Omega_j)$, $j \geqslant j_1$, and

$$\|\tilde{u} - \hat{u}_j\|_{\Omega, \infty} < \zeta(\Omega)/4, \qquad j \geqslant j_1. \tag{17}$$

By Assumption 2.2 we moreover have $A(0)-\tilde{u}(0)=A(\pi)-\tilde{u}(\pi)=0$ so that we can find a $\delta>0$ with

$$\|A-\tilde{u}\|_{\Omega \cap([0, \delta] \cup[\pi-\delta, \pi]), \infty}<\zeta(\Omega)/4. \tag{18}$$

Inequalities (17) and (18) together yield

$$\|A-\hat{u}_j\|_{\Omega \cap([0, \delta] \cup[\pi-\delta, \pi]), \infty}<\zeta(\Omega)/2 \leqslant \zeta(\Omega_j), \qquad j \geqslant j_1. \tag{19}$$

We next can select $j_2 \geqslant j_1$ such that $\varepsilon_j \leqslant \delta$ for all $j \geqslant j_2$. By the definitions of Ω_j and $\zeta(\Omega_j)$, inequalities (19) then imply

$$\|A-\hat{u}_j\|_{\Omega, \infty}=\zeta(\Omega_j), \qquad j \geqslant j_2, \tag{20}$$

so that the inequality signs in (13) can be replaced by equality signs for $j \geqslant j_2$. The latter shows that a solution $\hat{u}$ of $P[\Omega]$ in particular solves $P[\Omega_J]$ for some $J \geqslant j_2$. Thus, since $P[\Omega_J]$ has a unique solution, each solution of $P[\Omega]$ equals $\hat{u}$ on Ω_J. A solution of $P[\Omega]$ possesses a unique representation as a linear combination of $u_0, \ldots, u_{M-1}$ and therefore is uniquely determined by $\hat{u}$ also on Ω. Hence $\hat{u}$ is the only solution of $P[\Omega]$.

As an inference from Lemma 2.2, the solution $\hat{u}_J=\hat{u}$ of $P[\Omega_J]$ and $P[\Omega]$ is characterized by an alternant with points in Ω_J. By (19) and $\varepsilon_J \leqslant \delta$, this is also an alternant on Ω. Furthermore $\hat{u}_J=\hat{u}$ is a strongly unique solution of $P[\Omega_J]$ [2, p. 80] where, by (13) and (20), $\|A-\hat{u}\|_{\Omega_J, \infty}=\|A-\hat{u}\|_{\Omega, \infty}$ is given. Employing (15), we therefore can obtain

$$\begin{aligned} \|A-u\|_{\Omega, \infty} \geqslant \|A-u\|_{\Omega_J, \infty} & \geqslant \|A-\hat{u}\|_{\Omega_J, \infty}+\gamma\,\|\hat{u}-u\|_{\Omega_J, \infty} \\ & \geqslant \|A-\hat{u}\|_{\Omega, \infty}+(\gamma/\eta)\,\|\hat{u}-u\|_{\Omega, \infty}, \qquad u \in \mathcal{U}, \end{aligned}$$

with some constant $\gamma>0$. Thus $\hat{u}$ is a strongly unique solution of $P[\Omega]$. ∎

Theorem 2.2 guarantees all properties which are needed for the convergence proof of the second algorithm of Remez, inclusively the results on its rate of convergence [19].

Remark 2.2. Let $p=\infty$, let $\tilde{\Omega} \subseteq[0, \pi]$ be a nonempty closed set disjoint with Ω, let $U \in C(\tilde{\Omega})$ be a positive function, and define

$$K_M:=\left\{x \in \mathbb{R}^M \mid v(\omega)\left|\sum_{k=0}^{M-1} x_k c_k(\omega)\right| \leqslant U(\omega), \omega \in \tilde{\Omega}\right\}. \tag{21}$$

It is shown in [17, p. 203] that a design problem (10) constrained by (21) is equivalent to a problem of type (10) with $K_M:=\mathbb{R}^M$ and design frequency domain $\Omega \cup \tilde{\Omega}$, but only an a posteriori known weight function W. Hence Theorem 2.2 also applies to such a constrained problem.

The next theorem provides sufficient conditions for each $p \geqslant 1$ under which a solution of problem (10) is unique. In case $p = \infty$ these follow from the earlier results.

THEOREM 2.3. *Problem* (10) *possesses a unique solution for* (2)–(5) *when Assumptions* 2.1 *and* 2.2 *and the following are satisfied:*

(a) $p = 1$, $K_M := \mathbb{R}^M$, *and* Ω *is the union of nonempty closed intervals,*

(b) $1 < p < \infty$ *and* $K_M \subseteq \mathbb{R}^M$ *is nonempty, closed, and convex, or*

(c) $p = \infty$ *and* $K_M := \mathbb{R}^M$ *or* $K_M \subseteq \mathbb{R}^M$ *is defined as in Remark* 2.2.

Proof. The problem of approximating a continuous function on an interval $[a, b]$ by elements of a Haar space with respect to the L^1 norm has a unique solution. The reader may verify that this result (Jackson's unicity theorem in [2, p. 219]) remains true when the approximating space is a Haar space on $[a, b]$ and the approximation region is the union of finitely many closed intervals in $[a, b]$ instead. (Replace the integrals over $[x_{i-1}, x_i]$ in [2, p. 220] by integrals over $[x_{i-1}, x_i] \cap \Omega$ for that.) Therefore the unicity part of the proof of Theorem 2.2 is also valid for the L^1 norm instead of the maximum norm, as is easily checked. Thus (a) is proved.

Statement (b) follows from the facts that a closed convex set in a uniformly convex Banach space possesses exactly one point which has minimal distance to a given point [2, p. 22] and that the spaces $l^p(\Omega)$ and $L^p(\Omega)$ are uniformly convex for $1 < p < \infty$ (e.g., [7]). For $p = 2$, statement (b) also is a consequence of the projection theorem (e.g., [9]). Finally, assertion (c) is an implication of Theorem 2.2 and Remark 2.2. ∎

We note with respect to conditions (a) in the theorem that there exist no simple assumptions which guarantee uniqueness of a solution of a real linear minimum norm problem over a finite set when the l^1 norm is used. In particular the Haar condition is not sufficient then [31].

3. CONVERGENCE OF THE APPROXIMATION ERRORS

We next derive results on the convergence and order of convergence of the sequence of approximation errors $\{\lambda_{p,M}\}_{M \in \mathbb{N}}$ for $1 \leqslant p \leqslant \infty$. For that we need the following assumption.

ASSUMPTION 3.1. (i) *Ω is the union of disjoint closed intervals.*

(ii) *For some $M_0 \in \mathbb{N}$, let $K_M \subseteq \mathbb{R}^M$ be a nonempty closed set for all $M \geqslant M_0$.*

(iii) *The function A is once continuously differentiable on $[0, \omega_1)$ for some $\omega_1 > 0$ in case* (5) *if $0 \in \Omega_{\mathscr{D}}$ and once continuously differentiable on $(\omega_2, \pi]$ for some $\omega_2 < \pi$ in case* (3) *if $\pi \in \Omega_{\mathscr{D}}$.*

If Assumption 3.1 is fulfilled, we can define a function $\tilde{A}\colon \Omega_{\mathscr{D}} \to \mathbb{R}$ in the following way:

$$\tilde{A}(\omega) := A(\omega), \qquad \omega \in \Omega_{\mathscr{D}}, \qquad \text{for (2) and (4)},$$

$$\tilde{A}(\omega) := \begin{cases} \dfrac{A(\omega)}{\cos(\omega/2)}, & \omega \in \Omega_{\mathscr{D}} \setminus \{\pi\} \\ -2A'(\pi), & \omega := \pi \end{cases} \qquad \text{for (3)},$$

$$\tilde{A}(\omega) := \begin{cases} \dfrac{A(\omega)}{\sin(\omega/2)}, & \omega \in \Omega_{\mathscr{D}} \setminus \{0\} \\ 2A'(0), & \omega := 0 \end{cases} \qquad \text{for (5)}.$$

Application of de l'Hospital's rule in the cases of (3) and (5) shows that, under the additional Assumption 2.2, $\tilde{A}$ is continuous on $\Omega_{\mathscr{D}}$ for all four filter types. For $\delta > 0$ and $B \in C(\Gamma)$ we moreover let

$$\varrho_B(\Gamma, \delta) := \sup |B(\omega_1) - B(\omega_2)| \qquad \text{s.t.} \quad \omega_1, \omega_2 \in \Gamma, \quad |\omega_1 - \omega_2| \leqslant \delta,$$

be the *modulus of continuity* of B, where Γ is assumed to be the union of nonempty closed intervals. We note that $\varrho_B(\Gamma, \delta) \to 0$ is true for $\delta \to 0$ [3].

We first consider the sequence of approximation errors in problem (10) for the unconstrained case.

THEOREM 3.1. *Let $1 \leqslant p \leqslant \infty$ and $K_M := \mathbb{R}^M$ for all $M \in \mathbb{N}$. Let Assumptions 2.2 and 3.1 be satisfied. Then, for (2)–(5), there exist constants σ_1, σ_2, and σ_3 such that*

(a) $\lambda_{p,M} \leqslant \sigma_1[\varrho_{\tilde{A}}(\Omega, 1/M) + 1/M]$,

(b) $\lambda_{p,M} \leqslant \sigma_2/M$ *when $\tilde{A}$ is Lipschitz continuous on Ω,*

(c) $\lambda_{p,M} \leqslant \sigma_3/M^k$ *if $\Omega := [0, \pi]$ and $\tilde{A}^{(k)} \in C([0, \pi])$, where, for (5), $\lambda_{p,M}$ can be replaced by $\lambda_{p,M-1}$ in each case.*

Proof. Let $1 \leqslant p \leqslant \infty$ and $K_M := \mathbb{R}^M$ and define

$$\kappa_{\infty,M} := \min_{x \in \mathbb{R}^M} \max_{\omega \in \Omega} \left| \tilde{A}(\omega) - \sum_{k=0}^{M-1} x_k \cos(k\omega) \right|,$$

where we first consider the cases (2), (3), and (5). Using that the norms in (7) satisfy

$$1 \leqslant p \leqslant q \leqslant \infty \Rightarrow \|f\|_{\Omega,p} \leqslant (L_\Omega)^{1/p - 1/q} \|f\|_{\Omega,q}, \tag{22}$$

where L_Ω is the total length of intervals defining Ω, we obtain

$$\lambda_{p,M} \leqslant (L_\Omega)^{1/p} \lambda_{\infty,M} \leqslant (L_\Omega)^{1/p} \max_{\omega \in \Omega} [v(\omega) W(\omega)] \kappa_{\infty,M}. \tag{23}$$

By assumption we have $\Omega := \bigcup_{j=1}^m [a_j, b_j]$ with $a_j < b_j$, $j = 1, \ldots, m$. Let $\tilde{A}_e \in C([-\pi, \pi])$ be the function which coincides with $\tilde{A}$ on Ω, which for $m > 1$ equals the line interpolating $\tilde{A}(b_j)$ and $\tilde{A}(a_{j+1})$ on the interval (b_j, a_{j+1}), $j = 1, \ldots, m-1$, and which is defined on $[-\pi, 0]$ by $\tilde{A}_e(-\omega) := \tilde{A}_e(\omega)$, $\omega \in [0, \pi]$. Since $\tilde{A}_e$ is even, we obtain

$$\begin{aligned}\delta_{\infty,M} &:= \min_{x \in \mathbb{R}^{2M-1}} \max_{\omega \in [-\pi,\pi]} \left| \tilde{A}_e(\omega) - \sum_{k=0}^{M-1} x_k \cos(k\omega) - \sum_{k=1}^{M-1} x_{M-1+k} \sin(k\omega) \right| \\ &= \min_{x \in \mathbb{R}^M} \max_{\omega \in [-\pi,\pi]} \left| \tilde{A}_e(\omega) - \sum_{k=0}^{M-1} x_k \cos(k\omega) \right| \end{aligned} \tag{24}$$

[2, pp. 147–148]. By (23), this implies $\lambda_{p,M} \leqslant \sigma_0 \kappa_{\infty,M} \leqslant \sigma_0 \delta_{\infty,M}$ with some constant $\sigma_0 > 0$. Moreover, by construction of $\tilde{A}_e$, we get

$$\varrho_{\tilde{A}_e}\left([-\pi, \pi], \frac{\pi}{M}\right) = \varrho_{\tilde{A}_e}\left([0, \pi], \frac{\pi}{M}\right) \leqslant (\pi + 1)\, \varrho_{\tilde{A}_e}\left([0, \pi], \frac{1}{M}\right)$$

(see, e.g., [3, p. 41] for the inequality) and, by use of the triangle inequality,

$$\varrho_{\tilde{A}_e}\left([0, \pi], \frac{1}{M}\right) \leqslant \varrho_{\tilde{A}}\left(\Omega, \frac{1}{M}\right) + \frac{1}{M} \max_{1 \leqslant j \leqslant m-1} \left| \frac{\tilde{A}(a_{j+1}) - \tilde{A}(b_j)}{a_{j+1} - b_j} \right|.$$

Thus statements (a) and (c) follow from Jackson's Theorems III and IV, respectively [2, pp. 144–145]. Obviously (a) implies (b).

In case (4) one has

$$\sin(\omega) \sum_{k=0}^{M-1} x_k \cos(k\omega) = \sum_{k=1}^{M} y_k \sin(k\omega) \tag{25}$$

with some $y \in \mathbb{R}^M$ (e.g., [21]). Assumption 2.2 then implies $\tilde{A}(0) = A(0) = 0$ so that $\tilde{A}_e \in C([-\pi, \pi])$ can be defined analogously to the above definition as an odd function. Thus, letting here

$$\kappa_{\infty,M} := \min_{x \in \mathbb{R}^M} \max_{\omega \in \Omega} \left| \tilde{A}(\omega) - \sum_{k=1}^{M} x_k \sin(k\omega) \right|,$$

we can follow the first part of the proof where (23) holds with $v \equiv 1$ on Ω and the inequality $\kappa_{\infty,M} \leqslant \delta_{\infty,M+1}$ has to be used, with $\delta_{\infty,M}$ given by the first equation in (24). ∎

Theorem 3.1 shows that the sequence $\{\lambda_{p,M}\}$ of approximation errors tends to zero for $M \to \infty$ and that fast convergence can be expected if the function $\tilde{A}$ is sufficiently smooth. For practical purposes, it would of course be desirable that one could predict the size of M which, for a given function A, ensures a certain size of $\lambda_{p,M}$. However, the constants σ_1, σ_2, and σ_3 in the theorem, though they could be specified by the Jackson theorems used in its proof, will in general be too large for such a purpose since the constants in the Jackson theorems are generic constants which are valid for all 2π-periodic functions of a certain function space and relate to the worst possible bound for these. (Consider, for example, the function $\tilde{A}_e(\omega) := \cos(\omega)$ for which one obviously has $\delta_{\infty,2} = 0$ for $\delta_{\infty,M}$ as in (24). If one would not know the latter and would employ Jackson's Theorem II from [2, p. 143], in order to determine M such that $\delta_{\infty,2} \leqslant 10^{-4}$ holds true, one would obtain $\pi/(2M) \leqslant 10^{-4}$ resp. $M \geqslant 15708$.)

We finally wish to prove that the sequence of approximation errors $\{\lambda_{p,M}\}_{M \geqslant M_0}$ also converges to zero when constraints are involved, i.e., when one has $K_M \subseteq \mathbb{R}^M$ for $M \geqslant M_0$. It is clear that this can only occur when A is reachable by functions $v \sum_{k=0}^{M-1} x_k c_k$ with parameters x from K_M for $M \to \infty$. Therefore we need the following assumption on the sets K_M in problem (10) for given v.

Assumption 3.2. *Let the space $C(\Omega_{\mathscr{D}})$ be equipped with the maximum norm $\|\cdot\|_{\Omega_{\mathscr{D}},\infty}$. Each set K_M, $M \geqslant M_0$, is of the form $K_M := \{x \in \mathbb{R}^M \mid v \sum_{k=0}^{M-1} x_k c_k \in \mathscr{K}\}$ where $\mathscr{K}$ is a nonempty closed subset of $C(\Omega_{\mathscr{D}})$ which contains A in its interior $\mathscr{K}^\circ$.*

Note that this assumption relates to the normal situation in filter design that the parameter set in problem (10) is not arbitrary, but originates from constraints on the approximating function $v \sum_{k=0}^{M-1} x_k c_k$.

Example 3.1. (i) For $K_M := \mathbb{R}^M$, $M \in \mathbb{N}$, Assumption 3.2 is fulfilled with $\mathscr{K} := C(\Omega_{\mathscr{D}})$ for each v as in (2)–(5).

(ii) Let v be given as in (2)–(5) and $1 \leqslant s \leqslant \infty$ where s can differ from p in problem (10). Moreover let K_M, $M \geqslant M_0$, be defined by

$$K_M := \left\{ x \in \mathbb{R}^M \;\middle|\; \left\| v \sum_{k=0}^{M-1} x_k c_k \right\|_{\tilde{\Omega},s} \leqslant U \right\} \neq \varnothing,$$

where $U > 0$ is a given constant and $\tilde{\Omega} \subseteq \Omega_{\mathscr{D}}$ is a nonempty closed set. If $\|A\|_{\tilde{\Omega},s} < U$ is true, Assumption 3.2 is satisfied with $\mathscr{K} := \{F \in C(\Omega_{\mathscr{D}}) \mid \|F\|_{\tilde{\Omega},s} \leqslant U)\}$. For that note that, by (22), each $z \in C(\Omega_{\mathscr{D}})$ with $\|A - z\|_{\Omega_{\mathscr{D}},\infty} < L^{-1/s}(U - \|A\|_{\tilde{\Omega},s})$ satisfies

$$\|z\|_{\tilde{\Omega},s} \leqslant \|A - z\|_{\tilde{\Omega},s} + \|A\|_{\tilde{\Omega},s} \leqslant L^{1/s} \|A - z\|_{\tilde{\Omega},\infty} + \|A\|_{\tilde{\Omega},s} < U.$$

Employing Theorem 3.1, we can now provide a convergence result for the approximation errors in the presence of constraints.

COROLLARY 3.1. *Let Assumptions* 2.2 *and* 3.1 *be satisfied and let* $1 \leqslant p \leqslant \infty$. *Moreover, let Assumption* 3.2 *be fulfilled for some* v *as in* (2)–(5). *Then, for this* v, *one has* $\lim_{M \to \infty} \lambda_{p,M} = 0$.

Proof. Let K_M, $M \geqslant M_0$, be as required in Assumption 3.2. Setting $\Omega := \Omega_{\mathscr{D}}$, we can conclude from Theorem 3.1(a) that there exist $x^M \in \mathbb{R}^M$, $M \geqslant M_0$, such that

$$\lim_{M \to \infty} \left\| W \left(A - v \sum_{k=0}^{M-1} x_k^M c_k \right) \right\|_{\Omega_{\mathscr{D}}, \infty} = 0.$$

Thus, recalling (22), we also have

$$\lim_{M \to \infty} \left\| A - v \sum_{k=0}^{M-1} x_k^M c_k \right\|_{\Omega_{\mathscr{D}}, \infty} = 0, \qquad \lim_{M \to \infty} \left\| W \left(A - v \sum_{k=0}^{M-1} x_k^M c_k \right) \right\|_{\Omega, p} = 0. \tag{26}$$

Since A is assumed to lie in $\mathscr{K}^\circ$, the first limit in (26) implies that, for all sufficiently large M, the function $v \sum_{k=0}^{M-1} x_k^M c_k$ is in $\mathscr{K}$ and hence, by Assumption 3.2, x^M is in K_M. ∎

Remark 3.1. If $\mathscr{K}$ in Assumption 3.2 also is convex and if there exists $A_0 \in \mathscr{K}^\circ$ which has the same properties as A required by Assumptions 2.2 and 3.1, then it suffices to assume that A is in $\mathscr{K}$ instead of $\mathscr{K}^\circ$. In this case, one can use that, for each $\varepsilon > 0$, one can find a function $A_\delta := A + \delta(A_0 - A)$ with some $\delta \in (0, 1]$ such that $\|A - A_\delta\|_{\Omega_{\mathscr{D}}, \infty} \leqslant \varepsilon$. The function A_δ has the same properties as A in Assumptions 2.2 and 3.1 and, by the convexity of $\mathscr{K}$, lies in $\mathscr{K}^\circ$. See [24, p. 2] for details and a related result permitting equality constraints.

4. FINAL REMARKS

As is well known, the solution of problem (10) for $p = 2$ and $K_M := \mathbb{R}^M$ can be obtained from the so-called normal equations (e.g., [9]). In case $p = \infty$ and $K_M := \mathbb{R}^M$, the second algorithm of Remez, certainly is a proper tool for the solution of (10) since it possesses a quadratic rate of convergence with respect to the approximation errors [30]. By Theorem 2.2 the convergence of this algorithm is guaranteed for every closed subset Ω of $[0, \pi]$. If especially Ω equals the union of closed intervals, it is in particular advisable, at least for large M, to solve the continuous approximation problem rather than a discretized version of the problem only, as, for example, the implementation [10] of the Parks–McClellan algorithm does. This follows from the fact that, in case of discretization, at least $10M$ (equidistant) points should be selected in Ω to reach sufficient accuracy (e.g., [13, p. 89]) and that therefore such an approach requires an unreasonably high number of function evaluations. Also, if $\lambda_M^{(j)} \leqslant \lambda_{\infty, M}$ is the (in all points same) deviation of the iterate $x^{(j)} \in \mathbb{R}^M$ at the current set of $M+1$ points of Ω in the Remez algorithm, the relative deviation of $\lambda_M^{(j)}$ with respect to the continuous approximation error $\lambda_{\infty, M}$ can be estimated in the continuous case (and not in the discretized one) by

$$0 \leqslant \frac{\lambda_{\infty, M} - \lambda_M^{(j)}}{\lambda_{\infty, M}} \tag{27}$$

$$\leqslant \frac{\max_{\omega \in \Omega} |W(\omega)(A(\omega) - v(\omega) \sum_{k=0}^{M-1} x_k^{(j)} c_k(\omega))| - \lambda_M^{(j)}}{\lambda_M^{(j)}}. \tag{28}$$

Expression (28) can be computed from the available data and converges to zero for $j \to \infty$.

The Remez algorithm in [14] and its extension to certain constrained problems in [6] are only able to solve special problems. In contrast to that, the method in [18] can solve convex SIP problems and hence in particular maximum norm and least-squares norm design problems for linear-phase and nonlinear-phase FIR filters which include (in)finitely many convex inequality and finitely many linear equality constraints. Moreover, the method in [18] becomes an exchange algorithm related to the Remez type algorithms in [6] and [14] when it is applied to problems which are solvable by these algorithms, and it therefore can be expected to have a similar convergence behavior (see [25]). Examples of maximum and least-squares norm designs of linear-phase filters with maximal length $M = 1000$, obtained by the algorithm in [18], can be found in [16].

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